VOLUME 80 SEPARATE No. 439

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

MAY, 1954

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

STATISTICAL THEORY OF DROUGHTS

by E. J. Gumbel

HYDRAULICS DIVISION

(Discussion open until September 1, 1954)

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings – Separate No. 290). For free copies of this Separate—describing style, content, and format—address the Manager, Technical Publications, ASCE.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

STATISTICAL THEORY OF DROUGHTS

E. J. Gumbel
Adjunct Professor, Columbia University, New York*

ABSTRACT

Droughts are analyzed by the asymptotic theory of smallest values of a limited statistical variate. The extremal probability paper used for floods is used for the logarithms of droughts. If the lower limit of the discharges is assumed to be zero the probability function of the droughts becomes a straight line. If the lower limit exceeds zero, the three parameters in the probability function are estimated by the method of moments. A given statistical criterion indicates whether or not the lower limit may be assumed to be zero. Observations on the droughts of 13 rivers analyzed by this procedure show a close conformance with the theory. Therefore, extrapolation is permissible if the basic conditions prevail.

1. Introduction

The asymptotic theory of extreme values taken from an unlimited initial distribution of the exponential type is now widely used in the analysis of floods [2, 3, 9, 10].** The fact that this theory as well as many others assumes no upper limit, although shocking to some engineers, has not led to major difficulties. The main difficulty encountered has been the occurrence in smaller rivers of extraordinary floods exceeding the magnitude expected for the given number of observations.

The theory of extreme values is also an appropriate tool for the analysis of droughts which are defined as the annual minima of discharges. Up to now, the theory of smallest values taken from an unlimited distribution of the exponential type has been used [12]. The droughts are plotted in decreasing order on probability paper designed for the theory of extreme values. However, this procedure leads to a sort of discontinuity. Instead of one line, two lines have to be used, one dealing with moderate, and the other with severe droughts. The first part is interpreted as if it did not belong to the extreme values proper, but still to the initial distribution. Only the latter part serves for extrapolation. This leads to a loss of about 37% of the information furnished by the observation.

In the following another asymptotic theory of smallest values is used. It takes into account the fundamental difference between floods and droughts. For floods the lower limit of the discharges is of no practical interest. It does not affect the theory because the same asymptotic probability of largest values holds, as long as the initial variate is positive or unlimited to the left. However, for droughts being smallest values, the lower limit is assumed to be

*Work done in part as Consultant to Stanford University under the sponsorship of the Office of Naval Research and in part under grant from the Higgins foundation.

**The numbers in brackets [ ] refer to the bibliography.

(1) zero, or (2) a small positive value ϵ. Under adequate climatologic conditions, the first assumption may hold for small streams. The second assumption must be introduced for large rivers.

To use the lower limit, consider the three asymptotic probabilities of extreme values given in Table I where $\Phi(x)$ (and $P(x)$) is the probability that a largest (smallest) value is equal to or smaller (larger) than x. The second asymptotic probability was constructed by Fréchet [4]. The other two types are due to Fisher and Tippett [5]. A systematic theory linking the three asymptotes to the corresponding types of initial distributions was given by R. von Mises [7].

Probability tables for the first asymptotic distribution of extreme values have recently been published by the National Bureau of Standards [8]. The first and the third asymptotic distributions of extreme values are related by a logarithmic transformation which will be used in the following.

All three types involve a location parameter u and a scale parameter $1/\alpha$. In addition the third (limited) type involves an upper limit ω or a lower limit ϵ. For $x = u$, all six probabilities are

(1.1) $$\Phi(u) = 1/e = P(u)$$

Up to now, the first type has been used exclusively. In the following, the third type is used. Its analytic properties are investigated and its averages and moments calculated. Then the first asymptotic probability of largest values (used for floods) and the third asymptotic probability of smallest values (used for droughts) are compared and their similarities noted. Two methods for estimating the parameters are given corresponding to the cases (1) $\epsilon = 0$ and (2) $\epsilon > 0$. Finally, the theory is compared to observed droughts.

2. Lower Limit Zero

Let x be a drought, measured in cubic feet per second (cfs) or cubic feet per second per square mile, or cubic meters per second. Then $P^{(3)}(x)$ is the probability of a drought exceeding x. Suppose the lower limit is zero, then this probability is, from Table I, dropping the index (3)

(2.1) $$P(x) = \exp\left[-(x/u)^{\alpha}\right]$$

where $P(O) = 1$ means that all droughts exceed the discharge zero. The value $x = u$, the drought which will be exceeded 36.788% of the time may serve to characterize a given river. Therefore it is called the <u>characteristic drought.</u> The median drought $\check{x}$ is

(2.2) $$\check{x} = u\,(\lg 2)^{1/\alpha}$$

where lg stands for the natural logarithm. The modal drought $\tilde{x}$ obtained after two differentiations of (2.1).

(2.3) $$\tilde{x} = u(1 - 1/\alpha)^{1/\alpha}$$

is smaller than the characteristic drought u. A mode exists only for $1/\alpha < 1$, and the mode

(2.4) $$\left.\begin{matrix}\text{precedes}\\ \text{equals}\\ \text{exceeds}\end{matrix}\right\}\ \text{the median if}\ 1/\alpha \left\{\begin{matrix}>\\ =\\ <\end{matrix}\right.\ 0.30685$$

Hence, the existence of a mode and its location relative to the median depend on the scale parameter $1/\alpha$. In the case $1/\alpha = 0.30685$ the distribution is nearly symmetrical but not normal.

Table I

ASYMPTOTIC PROBABILITIES OF EXTREME VALUES

Initial Type	Largest Values	Conditions	Smallest Values	Conditions
1) Exponential	$\phi^{(1)}(x) = \exp\left[-e^{-\alpha(x-u)}\right];$	$\alpha > 0$	$p^{(1)}(x) = \exp\left[-e^{\alpha(x-u)}\right]$	$\alpha > 0$
2) Cauchy	$\phi^{(2)}(x) = \exp\left[-\left(\frac{u}{x}\right)^{\alpha}\right];$	$x \geqq 0;\ \alpha > 0,\ u > 0$	$p^{(2)}(x) = \exp\left[-\left(\frac{u}{x}\right)^{\alpha}\right]$	$x \leqq 0;\ \alpha > 0;\ u \leqq 0$
3) Limited	$\phi^{(3)}(x) = \exp\left[-\left(\frac{\omega - x}{\omega - u}\right)^{\alpha}\right];$	$x \leqq \omega,\ \alpha > 0,\ u < \omega$	$p^{(3)}(x) = \exp\left[-\left(\frac{x-\epsilon}{u-\epsilon}\right)^{\alpha}\right]$	$x \geqq \epsilon \cdot u \geqq \epsilon \cdot$ $\alpha > 0$

For numerical use of (2.1), the transformation

(2.5) $$x = e^{z};\ u = e^{v}$$

is introduced. Then the probability $\Pi(z)$ that the new variable exceeds z becomes

(2.6) $$\Pi(z) = \exp\left[-e^{\alpha(z-v)}\right]$$

Table I shows that this expression is the first asymptotic probability of smallest values and depends only upon the reduced variate

(2.7) $$-y = \alpha' \ (\log x - \log u)$$

Here, log stands for the common logarithm, and α and α' are related by

(2.7') $$\alpha' = 2.30259\,\alpha;\ 1/\alpha' = 0.43429/\alpha$$

The reduced variate y is a linear function of log x. Therefore the common logarithms of droughts can be treated by the methods used for floods. In particular, extremal probability paper can also be used for droughts if their common logarithms are plotted on the linear (vertical) scale. The common logarithm of the mth drought (in decreasing order) is plotted at the frequency $m/(N+1)$ where N stands for the total number of observed years. If the same drought x is observed k times, say, a mean rank

(2.8) $$\tilde{m} = \sqrt{m\,(m+k-1)}$$

is assigned to x and is then plotted at $\tilde{m}/(N+1)$. This procedure is used for all droughts except the smallest. If the smallest is observed several times, all values are plotted since the number of observations should be preserved on the graph. Graphs 1 and 2 show droughts of the Colorado and several other rivers traced on logarithmic extremal probability paper, prepared by the University of Michigan's School of Public Health. To facilitate reading, the probabilities P(x) and return periods T(x) are traced on horizontal parallel scales. The return period T(x) of a drought more severe than x is defined as usually by

$$T(x) = 1/[1 - P(x)]$$

whence from (2.1)

(2.9) $$T(x) = 1/\left(1 - \exp\left[-(x/u)^{\alpha}\right]\right)$$

The droughts decrease in amount and increase in severity with increasing return periods.

The procedure for droughts is thus strictly analogous to the procedure for floods. The interpretation and the influence of the parameters, however, differ in the two cases. For floods, the mode is u and exists independently of the value of the other parameter. The mode precedes the median and the skewness is fixed. For droughts the mode and the median given by (2.3) and (2.2) and the skewness depend upon $1/\alpha$. For floods this parameter has the dimension of a discharge; for droughts it is dimensionless. Therefore the slope of the line (2.7) is independent of the measures chosen for the droughts.

Consider now the asymptotic behavior of severe droughts. As shown previously [6] the natural logarithm of the return period traced on extremal probability paper converges toward the reduced variate y. Therefore the relation (2.9) can be written asymptotically for severe droughts

(2.9') $$-\lg T \sim \alpha(\lg x - \lg u)$$

and log x converges to a linear function of log T while, for floods, x itself converges to a linear function of log T. The parameter $1/\alpha$ is the negative of the derivative of the logarithm of the drought with respect to the logarithm of the return period. Therefore droughts decrease asymptotically as a power of the return period, while floods increase as the logarithm of the return period. The return period of the zero drought is infinity. Consequently, the assumption of a lower drought limit of zero does not mean that a river may be expected to be dry at a certain date.

The drought x_T as a function of the return period T may be written asymptotically, from (2.9') for severe droughts, as

(2.10)
$$\frac{x_T}{u} \sim \left(\frac{1}{T}\right)^{1/\alpha}$$

Therefore the drought x_{2T} corresponding to the return period 2T is related to the drought x_T by

(2.11)
$$x_{2T} \sim x_T\left(\frac{1}{2}\right)^{1/\alpha}$$

If return period is doubled, the expected severest drought is asymptotically the original drought divided by $2^{1/\alpha}$ while the doubling of the return period leads only to the addition of $(\lg 2)/\alpha$ to a flood.

Conversely, consider the return period T' during which a drought decreases to one half its initial amount. From

$$\frac{x_{T'}}{u} = \frac{x_T}{2u}$$

and (2.10) it follows that this return period T' is asymptotically 2^{α} T where T is the original return period. On the other hand the return period corresponding to the double of a given flood is asymptotically the square of the original return period multiplied by $e^{\alpha u}$.

3. Estimation of the parameters for $\epsilon = 0$

Since the graphical procedure is based on the logarithms of the droughts, it is appropriate to use them also for estimating the parameters. This means that, instead of estimating u, we estimate log u and $1/\alpha'$ from the linear relation (2.7). The method of least squares can be used to minimize either the horizontal or vertical distances between theory and observations. The following compromise between the two methods, worked out for climatic extremes [1] and for the linear relation between the extreme x and the reduced extreme y permits a rapid calculation and has been shown to be quite reliable. According to this method, the parameter $1/\alpha'$ is estimated by

(3.1)
$$\frac{1}{\alpha'} = \frac{s(\log x)}{\sigma_N}$$

and log u is estimated from

(3.2)
$$\log u = \overline{\log x} + \bar{y}_N/\alpha'$$

This procedure requires the calculation of the mean $\overline{\log x}$ and of the standard deviation s(log x) of the logarithms. The reduced mean $\bar{y}_N$ and the reduced standard deviation σ_N depending only upon the sample size N are given in Table II.

TABLE II

Expected means $\bar{y}_N$ and standard deviations σ_N of reduced extremes

N	$\bar{y}_N$	σ_N	N	$\bar{y}_N$	σ_N
15	.5128	1.0206	48	.5477	1.1574
16	57	316	49	81	590
17	81	411	50	85	607
18	.5202	493	51	89	623
19	20	565	52	93	638
20	36	628	53	97	653
21	52	696	54	.5501	667
22	68	754	55	04	681
23	83	811	56	08	696
24	96	864	57	11	708
25	.5309	915	58	15	721
26	20	961	59	18	734
27	32	1.1004	60	21	747
28	43	047	62	27	770
29	53	086	64	33	793
30	62	124	66	38	814
31	71	159	68	43	834
32	80	193	70	48	854
33	88	226	72	52	873
34	96	255	74	57	890
35	.5403	285	76	61	906
36	10	313	78	65	923
37	18	339	80	69	938
38	24	363	82	72	953
39	30	388	84	76	967
40	36	413	86	80	980
41	42	436	88	83	994
42	48	458	90	86	1.2007
43	53	480	92	89	020
44	58	499	94	92	032
45	63	519	96	95	044
46	68	538	98	98	055
47	73	557	100	.5600	065

Finally the theoretical values of the droughts are obtained from (2.7) as

(3.3) $$\log x = \log u - y/\alpha'$$

If the probability paper is scaled in logarithms, the theoretical droughts can be immediately read off the straight line (3.3). In particular, we obtain the 100 years drought by introducing the corresponding value of the reduced variate y = +4.6, and the expected drought for any other desired return period by extending the line (3.3).

It is interesting to compare two rivers such that

$$u_1 > u_2 \; ; \frac{1}{\alpha_1} > \frac{1}{\alpha_2}$$

From (3.3) it follows for y = +4.6 that the first river which has larger discharges may have a more severe hundred years drought, i.e., a smaller value than the second river which has smaller discharges. This is not surprising since a river with a smaller modal flood may also have a higher 100 years flood than a second river with a higher modal flood, provided that the dispersion for the first river is larger than for the second one.

Table III shows the calculation of the parameters u and $1/\alpha$ in the theoretical line (3.3) for the Colorado River at Lees Ferry (Ariz.), 1922-39, (W.S.P. 879); Arkansas River, at Syracuse, Kansas, 1922-1939 (W.S.P. 877); Sta. Clara River near Central, Utah, 1911-30 (W.S.P. 879); Bourbeuse River at Union, Montana, 1922-39 (W.S.P. 877); Swift River at West Ware, Mass., 1913-1945 (W.S.P. 1105); Colorado River at Glenwood Springs, 1904-09, 1912-39 (W.S.P. 879); Chippewa River at Jini Falls Dam, Wisconsin, 1925-48 and the Colorado River, Bright Angel Creek, 1924-39 (W.S.P. 879). The observations and the results of the linear theory (3.3) are shown in graphs 1 and 2.

4. The general case

The observed droughts traced in Graphs 1 and 2 are closely scattered about the theoretical lines (3.3). However, in other cases - see Graphs 3 and 4 - the plotted points approach a curve which falls off less rapidly for increasing return periods and the assumption $\epsilon = o$ is too strong. To account for these cases, we use the asymptotic probability P(x) that the smallest value exceeds x where the initial variate is limited by $x \geqq \epsilon > 0$. From Table I

(4.1) $$P(x) = \exp\left[-\left(\frac{x-\epsilon}{u-\epsilon}\right)^{\alpha}\right]$$

with

(4.1') $$P(u) = 1/e \; ; \; P(\epsilon) = 1$$

Thus u and ϵ are location parameters and $1/\alpha$ a scale parameter. The relation between mode and median is the same as in the special case. Therefore a pseudo-symmetrical case exists here too.

For the graphical representation, the transformation (2.7) is used in the form

(4.2) $$\log(x-\epsilon) = \log(u-\epsilon) - y/\alpha'$$

The observed droughts are again plotted on a decreasing scale on logarithmic extremal probability paper. However, the relation between the reduced variate

TABLE III

Calculation of the Parameters for Different Rivers

River			Colorado	Arkansas	Santa Clara	Bourbeuse	Swift	Colorado	Chippewa	Colorado
Locality			Lees Ferry (Ariz.)	Syracuse (Kan.)	Central (Utah)	Union (Mont.)	West Ware (Mass.)	Glenwood Springs (Colo.)	Jini Falls (Wisc.)	Bright Angel Creek (Ariz.)
Publication			W.S.P. 879	W.S.P. 877	W.S.P. 879	W.S.P. 877	W.S.P. 1105	W.S.P. 879		W.S.P. 879
Period			1922-39	1922-39	1911-30	1922-39	1913-45	1904-10 1912-39	1925-48	1924-39
Number of years	Observation	N	= 18	18	20	18	33	35	24	16
Mean logarithm	Observation	$\overline{\log x}$	= 3.38528	.42067	.76535	1.4680	1.65887	2.66444	3.28992	3.42324
Standard deviation of logarithms	Observation	$s(\log x)$	= 0.17712	.27365	.16478	.15227	.22974	.08662	.16908	.16334
Parameter of scale	Equ. (3.1)	$1/\alpha'$	= 0.16879	.26080	.15504	.14511	.20465	.07675	.15562	.15834
Parameter of location	Equ. (3.2)	$\log u$	= 3.47307	.55633	.84653	1.5435	1.76913	2.70591	3.37232	3.50489
Characteristic drought in c.f.s.	Equ. (3.2)	u	= 2972	3.60	7.02	34.95	58.8	508	2357	3198
Minimum in c.f.s.	Observation	x_N	= 1000	1.0000	3.00	14.0	15.0	286	871	990
100 years' drought in c.f.s.	Equ. (3.3)	x_{100}	= 497	.22730	1.36	7.52	6.7	230	453	483

y and log x is no longer linear. The second derivative $\frac{d^2(\log x)}{dy^2}$ is negative. Therefore the curves log x, y are bent downward. As long as x is large compared to ϵ, the curves are nearly linear. For increasing return periods T they approach a line parallel to the y axis at the asymptotic distance $x = \epsilon$. All curves with the same characteristic drought u intersect at the point $x = u$, $y = 0$. For fixed values of u and ϵ, the convergence toward the asymptotic value is more rapid as $1/\alpha$ increases. Two curves log x, y are parallel if they have the same values of $1/\alpha$ and $u - \epsilon$ although the parameters u and ϵ themselves may differ.

Since the previous graphical estimate of the parameters is no longer possible, the classical method of moments is used. In the case $\alpha = 1$, the probability function (4.1) degenerates into an exponential function. The characteristic drought u becomes equal to the mean, and the lower limit is estimated from the observed mean $\bar{x}$ and standard deviation s of the droughts, as

$$\hat{\epsilon} = \bar{x} - s$$

Here and in the following an estimated parameter is designated by the symbol $\hat{}$.

In general we need 3 moments. The distribution p(x) of droughts obtained from (4.1) is

$$p(x) = \frac{\alpha}{u-\epsilon}\left(\frac{x-\epsilon}{u-\epsilon}\right)^{\alpha - 1} \exp\left[-\left(\frac{x-\epsilon}{u-\epsilon}\right)^{\alpha}\right]$$

Consequently the reduced moments $\overline{(x- \quad)^k}/(u- \quad)^k$ of order k are the gamma functions

$$\overline{\left(\frac{x-\epsilon}{u-\epsilon}\right)^k} = \Gamma(1+1/k) \tag{4.3}$$

For $k = 1,2,3$, three equations are obtained for the three parameters

$$\overline{\frac{x-\epsilon}{u-\epsilon}} = \Gamma(1+1/\alpha); \quad \overline{\left(\frac{x-\epsilon}{u-\epsilon}\right)^2} = \Gamma(1+2/\alpha); \quad \overline{\left(\frac{x-\epsilon}{u-\epsilon}\right)^3} = \Gamma(1+3/\alpha) \tag{4.4}$$

The variance σ^2 and the third central moment μ_3 are

(4.4')

$$\sigma^2 = (u-\epsilon)^2\Big(\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\Big); \quad \mu_3 = \overline{(x-\epsilon)^3} - 3\overline{(x-\epsilon)^2}\cdot\overline{(x-\epsilon)} + 2\overline{(x-\epsilon)}^3$$

From (4.4) the skewness $\sqrt{\beta_1}$ defined by

$$\sqrt{\beta_1} = \mu_3\sigma^{-3} \tag{4.5}$$

becomes

(4.5')

$$\sqrt{\beta}_1 = \Big(\Gamma(1+3/\alpha) - 3\Gamma(1+2/\alpha)\Gamma(1+1/\alpha) + 2\Gamma^3(1+1/\alpha)\Big)\Big(\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\Big)^{-3/2}$$

This expression depends only upon the parameter $1/\alpha$. If the population value $\sqrt{\beta_1}$ is replaced by the sample value $\sqrt{b_1}$, an estimate of $1/\alpha$ is obtained. To facilitate this procedure, the right side of equation (4.5') as function of $1/\alpha$ is given in Table IV, cols. 4 and 1.* The value of $1/\alpha'$ in equation (4.2) is obtained from (2.7').

*This Table was calculated by Gladys R. Garabedian of Stanford University. I profit on this occasion to thank her for this important contribution.

The two remaining location parameters, the characteristic drought u and the lower limit ϵ, are simple to estimate. Equation (4.4) for the mean leads with the help of (4.4') to

$$(4.6) \qquad u = \bar{x} + \sigma A(\alpha)$$

where

$$(4.6') \qquad A(\alpha) = \Big(1 - \Gamma(1+1/\alpha)\Big)\Big(\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\Big)^{-1/2}$$

is given in Table IV, col. 3. Since $1/\alpha$ has already been estimated from (4.5'), the parameter u may be estimated from (4.6) after replacing the population mean and standard deviation by the sample values. The result can be checked from the graph of the droughts on extremal probability paper with the help of the first equation (4.1').

To estimate the lower limit ϵ, equation (4.4) for the first moment is written in the form

$$(4.7) \qquad \epsilon = \frac{\bar{x} - u\,\Gamma(1+1/\alpha)}{1 - \Gamma(1+1/\alpha)}$$

and the value of $\bar{x}$ is introduced from (4.6) and (4.6'). This leads, after trivial simplifications, to

$$(4.8) \qquad \epsilon = u - \sigma\, B(\alpha)$$

where

$$(4.8') \qquad B(\alpha) = \Big(\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\Big)^{-1/2}$$

is given in Table IV, col. 2. For estimating the lower limit ϵ we use the previous estimates of $1/\alpha$ and u, and replace the population value σ in (4.8) by the sample value s. It is worthwhile mentioning that the lower limit is estimated here directly without successive approximations.

Equations (4.6) and (4.8) lead to a criterion which decides whether or not zero should be chosen as the lower limit since

$$(4.9) \qquad \epsilon \geqq 0 \text{ if } \bar{x} + s\Big(A(\alpha) - B(\alpha)\Big) \geqq 0$$

This condition may be written from (4.6') and (4.8')

$$\bar{x}/s \geqq \Gamma(1+1/\alpha)\Big(\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\Big)^{-1/2}$$

whence

$$(4.9') \qquad \epsilon \geqq 0 \text{ if } \overline{x_2}/\bar{x}^2 \leq \Gamma(1+2/\alpha)/\Gamma^2(1+1/\alpha)$$

The lower limit is taken to be zero if the equality is fulfilled within the errors of random sampling. If ϵ is negative and small, it can safely be assumed to be zero. The theory fails if the lower limit turns out to be larger than the observed smallest drought - provided this observation is reliable - and if ϵ takes on a large negative value.

In the pseudo-symmetrical case, $1/\alpha = 0.30685$ and the other two parameters are estimated using (4.6) and (4.8) as

$$\hat{u} = \bar{x} + 0.34230\,s \;:\; \hat{\epsilon} = u - 3.30649\,s$$

The estimation of the three parameters is shown in detail in Table V for the Connecticut River, and in Table VI for other rivers. The three parameters are introduced into equation (4.2) which leads to $\log(x - \hat{\epsilon})$ and hence to the theoretical values log x and x as functions of the reduced variate y, i.e., of the return period T.

TABLE IV

Estimation of the Three Parameters

Scale parameter	Multiple of Standard Deviations for lower limit ϵ	for parameter u	Reduced third moment
$1/\alpha$	$B(\alpha)$	$A(\alpha)$	$\sqrt{\beta_1}$
	Equ. (4.8')	Equ. (4.6')	Equ. (4.5')
.01	78.981714	.448154	-1.081272
.02	39.989044	.446110	-1.024853
.03	26.986212	.443925	- .970702
.04	20.480808	.441603	- .918459
.05	16.574350	.439150	- .867967
.06	13.967343	.436568	- .819101
.07	12.102862	.433863	- .771740
.08	10.702446	.431038	- .725772
.09	9.611395	.428096	- .681102
.10	8.736889	.425043	- .637637
.11	8.019861	.421881	- .595296
.12	7.420934	.418614	- .554002
.13	6.912848	.415245	- .513687
.14	6.476131	.411778	- .474287
.15	6.096505	.408216	- .435743
.16	5.763261	.404563	- .398002
.17	5.468210	.400822	- .361012
.18	5.204984	.396996	- .324729
.19	4.968556	.393087	- .289108
.20	4.754903	.389100	- .254110
.21	4.560770	.385036	- .219697
.22	4.383495	.380900	- .185835
.23	4.220878	.376693	- .152490
.24	4.071085	.372419	- .119634
.25	3.932577	.368079	- .087237
.26	3.804052	.363678	- .055272
.27	3.684400	.359218	- .023715
.28	3.572672	.354700	+ .007458
.29	3.468048	.350129	.038270
.30	3.369818	.345505	.068742
.31	3.277364	.340832	.098893
.32	3.190146	.336112	.128743
.33	3.107688	.331348	.158308
.34	3.029573	.326541	.187606
.35	2.955428	.321694	.216653
.36	2.884924	.316809	.245465
.37	2.817768	.311889	.274055
.38	2.753697	.306935	.302437
.39	2.692475	.301949	.330625
.40	2.633890	.296935	.358632
.41	2.577752	.291893	.386468
.42	2.523887	.286825	.414147
.43	2.472138	.281734	.441678
.44	2.422364	.276622	.469072
.45	2.374435	.271490	.496340
.46	2.328232	.266340	.523491
.47	2.283647	.261174	.550535
.48	2.240583	.255993	.577481
.49	2.198946	.250801	.604336
.50	2.158655	.245597	.631111
.51	2.119632	.240384	.657812
.52	2.081807	.235163	.684448
.53	2.045114	.229937	.711026
.54	2.009492	.224706	.737553
.55	1.974885	.219472	.764038

Scale parameter	Multiple of Standard Deviations for lower limit ϵ	for parameter u	Reduced third moment
$1/\alpha$	$B(\alpha)$	$A(\alpha)$	$\sqrt{\beta_1}$
.56	1.941242	.214237	.790486
.57	1.908514	.209002	.816904
.58	1.876656	.203768	.843299
.59	1.845626	.198537	.869677
.60	1.815385	.193311	.896045
.61	1.785897	.188090	.922408
.62	1.757128	.182875	.948772
.63	1.729045	.177669	.975143
.64	1.701620	.172473	1.001527
.65	1.674824	.167287	1.027929
.66	1.348631	.162113	1.054354
.67	1.623017	.156951	1.080808
.68	1.597958	.151804	1.107296
.69	1.573432	.146672	1.133822
.70	1.549420	.141557	1.160393
.71	1.525901	.136459	1.187012
.72	1.502857	.131379	1.213685
.73	1.480272	.126318	1.240415
.74	1.458130	.121278	1.267209
.75	1.436413	.116260	1.294070
.76	1.415109	.111263	1.321004
.77	1.394204	.106290	1.348013
.78	1.373683	.101340	1.375104
.79	1.353536	.096416	1.402280
.80	1.333750	.091517	1.429545
.81	1.314314	.086645	1.456904
.82	1.295217	.081799	1.484362
.83	1.276450	.076982	1.511921
.84	1.258002	.072194	1.539587
.85	1.239865	.067435	1.567363
.86	1.222031	.062706	1.595254
.87	1.204489	.058008	1.623264
.88	1.187234	.053341	1.651396
.89	1.170256	.048707	1.679655
.90	1.153550	.044105	1.708045
.91	1.137107	.039536	1.736570
.92	1.120922	.035002	1.765232
.93	1.104988	.030501	1.794038
.94	1.089299	.026035	1.822990
.95	1.073849	.021605	1.852093
.96	1.058632	.017211	1.881350
.97	1.043644	.012853	1.910765
.98	1.028880	.008531	1.940343
.99	1.014333	.004247	1.970086
1.00	1.0	0.0	2.0
1.0	1.	0.	2.
1.1	.867491	- .040326	2.309348
1.2	.752233	- .076579	2.640035
1.3	.651524	- .108617	2.996146
1.4	.563330	- .136421	3.382013
1.5	.486053	- .160077	3.802311
1.6	.418382	- .179747	4.262142
1.7	.359209	- .195656	4.767125
1.8	.307573	- .208071	5.323478
1.9	.262625	- .217284	5.938118
2.	.223607	- .223607	6.618761
3.	.038236	- .191180	19.584859
4.	.005016	- .115370	60.091733
5.	.000526	- .062593	190.113240

TABLE V

Estimate of the Three Parameters, Connecticut River at Sunderland, (Mass.)

W.S.P. 1105

1	Mean drought	$\bar{x}$	from observation	1353.9
2	Mean square	$\overline{x^2}$	observation	1943682.143
3	Variance	s^2	from (1) and (2)	1105595.673
4	Standard deviation	s	from (3)	332.505
5	Third power	s^3	from (3) and (4)	36761607.56
6	Mean cubic	$\overline{x^3}$	observation	2945659892.
7	Third Central Moment	m_3	(1),(2),(6)	14673590.
8	Skewness	$\sqrt{b_1} = m_3 s^{-3}$	formula (4.5); (7),(5)	0.39915
9	Parameter	$1/\hat{\alpha}$	(8) Table IV, col. 1	0.41458
10	Parameter	$1/\hat{\alpha}'$	formula (2.7')	0.18005
11	Factor	$A(\alpha)$	(9) Table IV, col. 3	0.28957
12	Char. drought	$\hat{u}$	(1)(4)(11) formula (4.6)	1450.2
13	Factor	$B(\alpha)$	(9) Table IV, col. 2	2.55308
14	Product	$s\,B(\alpha) = \hat{u} - \hat{\epsilon}$	(4), (13)	848.9
15	Lower Limit	$\hat{\epsilon}$	(12),(14)	601.3
16	Logarithm	$\log(\hat{u} - \hat{\epsilon})$	(12),(15)	2.92886
17	Minimum	x_{28}	observation	780.
18	Hundred years drought	x_{100}	formula (4.2); $y = +4.6$	724.

TABLE VI

The 3 Parameters for Different Rivers

1	River	Rhein	Inn	Arkansas R.	Housatonic R.
2	Station	Rheinfelden	Martinsbruck	Salida (Colo.)	Falls Village (Conn.)
3	Period	1918-33; 35 - 1949	1917 - 1949	1912-18; '20-'39	1913-45
4	Publication	Schweiz. Hydrogr.	Jahrbucher	W.S.P. 877	W.S.P. 1105
5	Number of years N	31	33	27	33
6	Unit	cubic meter per second		c.f.s.	c.f.s.
7	Mean $\bar{x}$	437.19	12.709	185.70	87.939
8	Standard deviation s	77.620	1.50240	19.909	54.684
9	Skewness $\sqrt{b_1}$	.60618	.27294	.24634	1.3660
10	Parameter $1/\hat{\alpha}'$	.21309	.16052	.15636	.33730
11	Char. drought $\hat{u}$	456.63	13.178	192.01	93.572
12	Lower limit $\hat{\epsilon}$	287.98	8.9406	134.58	18.078
13	Observed Minimum x_N	306.00	10.200	155.00	24.000
14	100 years drought x_{100}	302.7	9.484	145.90	20.000

After calculation of the three parameters (lines 10-12) the theoretical droughts x obtained from (4.2) are plotted against y on logarithmic extremal probability paper (see graphs 3 and 4). The fit of the theory to the observations is highly satisfactory, and the expected droughts for the desired return periods can easily be read off the graphs.

5. Summary of procedures

The extremal probability paper which is used by the Geological Survey [2,9] for floods can also be used for droughts if the logarithms of the droughts are plotted in decreasing order of magnitude. If the series of points (log x, y) is scattered about a straight line the limiting value of the droughts may be assumed to be zero. In this case, the two parameters are estimated from (3.1) and (3.2) and the theoretical droughts are obtained from (3.3). A criterion for the validity of the assumption $\epsilon = 0$ is given in (4.9).

If the series of points (log x, y) falls off less rapidly with increasing return periods, the first three moments have to be calculated. Then the three parameters are estimated from equations (4.5') (4.6) (4.8) with the help of Table IV. The theoretical values of x calculated from equation (4.2) are traced as functions of y.

If the number of observations is large enough so that the sampling errors are small and if it is reasonable to assume that the basic conditions prevail, the theoretical curves thus obtained can be used to estimate the most severe drought expected within a given number of years. Further, it is believed that this procedure may be useful for solving storage and irrigation problems.

REFERENCES

[1.] Anonymous — Evaluation of climactic extremes. Report 175. Environmental protection section. Office of the Quartermaster General, Washington, 1951.

[2.] C. W. Carter — Floods in Georgia. Geological Survey Circular No. 100, Washington, 1951.

[3.] W. P. Cross — Floods in Ohio. Magnitude and frequency Bulletin No. 7, Ohio Water Resources Board, Columbus (Ohio) 1946.

[4.] M. Fréchet — Sur la loi de probabilité de l'écart maximum, Annales de la Société Polonaise de Mathématique, Vol. 6, Cracovie, 1925.

[5.] R. A. Fisher and L. H. C. Tippett — Limiting forms of the frequency distribution of the largest or smallest member of a sample. Proc. Cambridge Philosophical Society, Vol. 24, 1926.

[6.] E. J. Gumbel — The return period of flood flows. Ann. Math. Statistics, Vol. 12, 1941.

[7.] R. von Mises — La distribution de la plus grande de n valeurs. Revue Math. l'Union inter-balcanique, Vol. 1. Athènes, 1936.

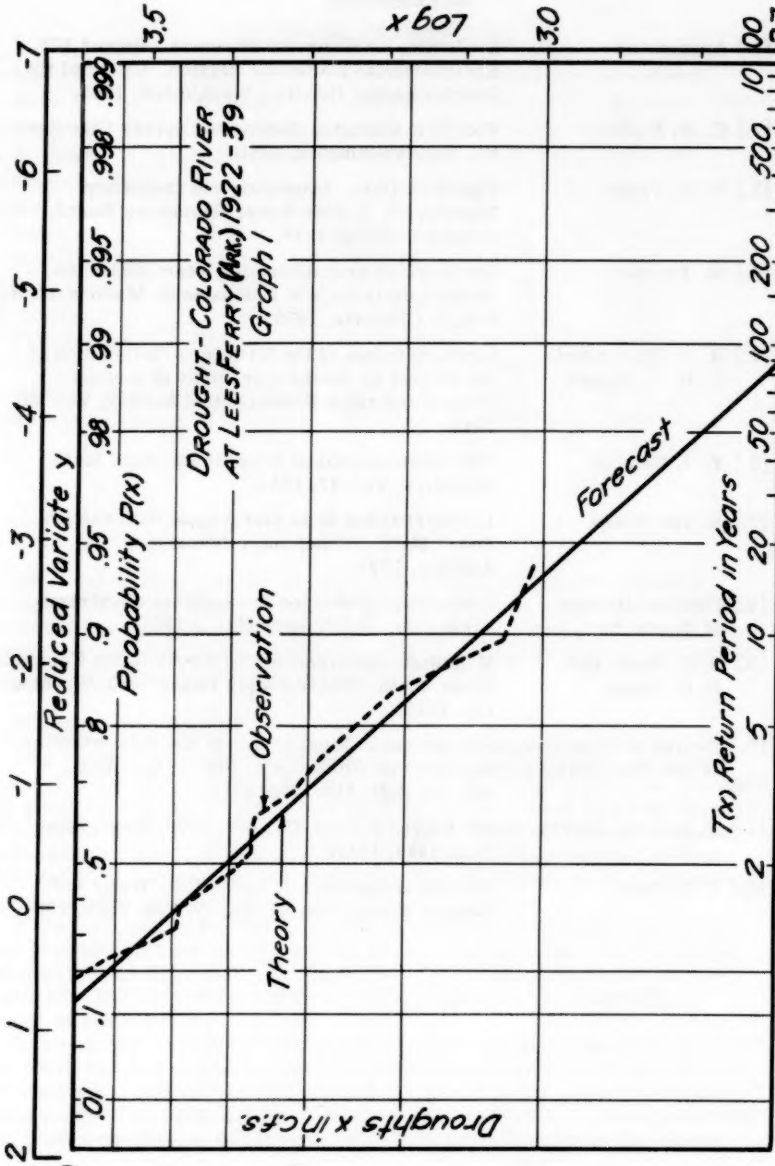
[8.] National Bureau of Standards — Probability tables for the analysis of extreme value data. Washington (D.C.) 1953.

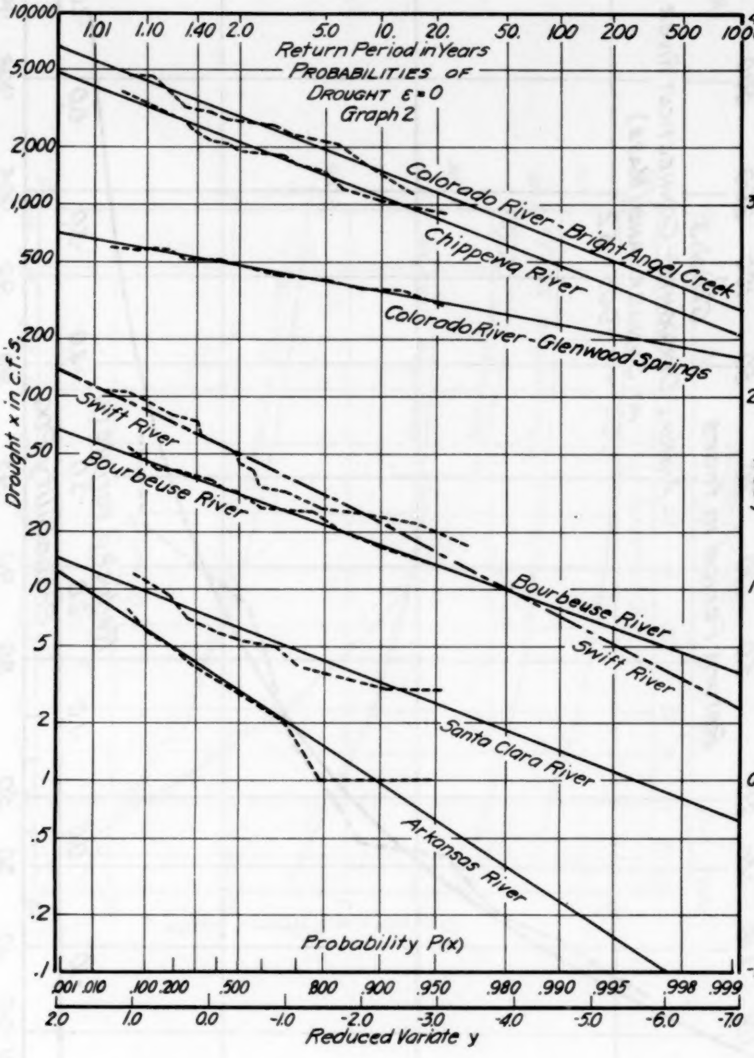
[9.] S. E. Rantz and H. C. Riggs — Magnitude and frequency of floods in the Columbia River basin. Water Supply Paper 1080, Washington, 1949.

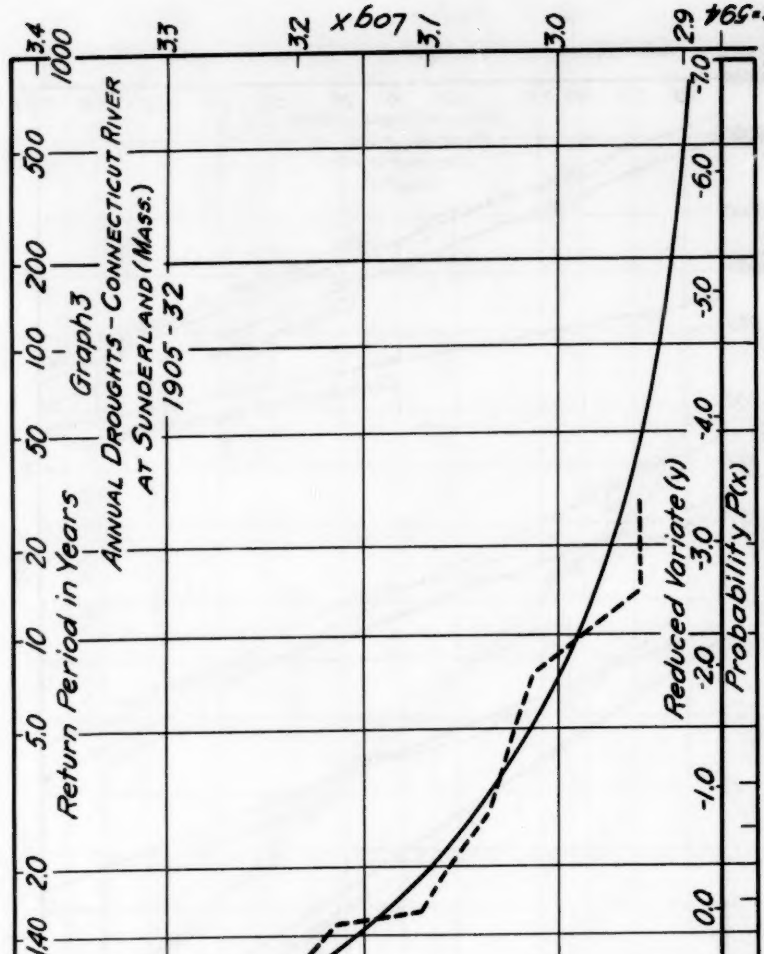
[10.] Review of flood frequency methods. Final report of the Subcommittee of the Joint Division Committee on Floods, Am. Soc. of Civ. Eng., vol. 77, Sept. 110, Dec. 1951.

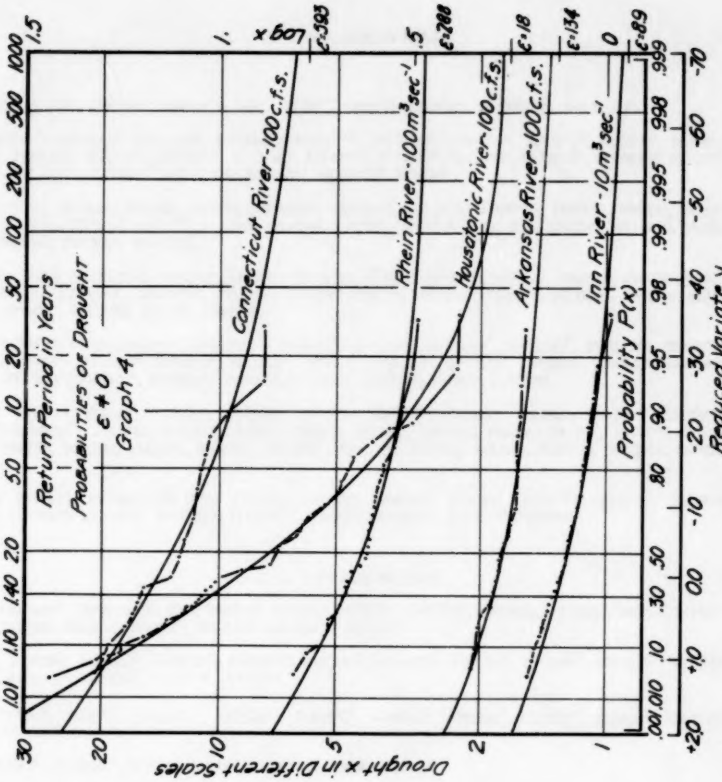
[11.] U. S. Geol. Survey, Water Supply Papers 877, 879, 1105, Washington, D.C. 1939, 1949.

[12.] C. J. Velz — Graphical approach to statistics. Water and Sewage Works, vol. 97, no. 10, New York, 1950.


Drought - Colorado River
at Lees Ferry (Ark.) 1922-39
Graph I
Reduced Variate y
Probability P(x)
Log x
T(x) Return Period in Years
Droughts x in c.f.s.
Observation
Theory
Forecast
-7
-6
-5
-4
-3
-2
-1
0
1
2
.01
.1
.5
.8
.9
.95
.98
.99
.995
.998
.999
2
5
10
20
50
100
200
500
1000
3.5
3.0
2.7
4000
3000
2000
1000
500


Return Period in Years
1.01 1.10 1.40 2.0 5.0 10. 20. 50. 100 200 500 1000
PROBABILITIES OF DROUGHT ε = 0
Graph 2
Colorado River - Bright Angel Creek
Chippewa River
Colorado River - Glenwood Springs
Swift River
Bourbeuse River
Bourbeuse River
Swift River
Santa Clara River
Arkansas River
Drought x in c.f.s.
10,000 5,000 2,000 1,000 500 200 100 50 20 10 5 2 1 .5 .2 .1
Log x
4 3 2 1 0 -1
Probability P(x)
.001 .010 .100 .200 .500 .800 .900 .950 .980 .990 .995 .998 .999
2.0 1.0 0.0 -1.0 -2.0 -3.0 -4.0 -5.0 -6.0 -7.0
Reduced Variate y


Graph 3
Annual Droughts - Connecticut River
at Sunderland (Mass.)
1905-32
Return Period in Years
1.01 1.10 1.40 2.0 5.0 10 20 50 100 200 500 1000
Log x
3.4 3.3 3.2 3.1 3.0 2.9
Droughts x
2400 2000 1500 1000 500
Reduced Variate (y)
-2.0 +1.0 0.0 -1.0 -2.0 -3.0 -4.0 -5.0 -6.0 -7.0
Probability P(x)
.001 .010 .10 .30 .50 .80 .90 .95 .98 .99 .995 .998 .999
E-594


Return Period in Years
PROBABILITIES OF DROUGHT
ε ≠ 0
Graph 4
Connecticut River - 100 c.f.s.
Rhein River - 100 m³sec⁻¹
Housatonic River - 100 c.f.s.
Arkansas River - 100 c.f.s.
Inn River - 10 m³sec⁻¹
Log x
ε=.393
ε=.288
ε=.18
ε=.134
ε=.89
Probability P(x)
Reduced Variate y
Drought x in Different Scales
1.01 1.10 1.40 2.0 5.0 10 20 50 100 200 500 1000
.001 .010 .10 .30 .50 .80 .90 .95 .98 .99 .995 .998 .999
+2.0 +1.0 0.0 -1.0 -2.0 -3.0 -4.0 -5.0 -6.0 -7.0
30 20 10 5 2 1

PROCEEDINGS-SEPARATES

The technical papers published in the past twelve months are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Separate Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

VOLUME 79 (1953)

JUNE: 194(CP) & (AT), 195(SM), 196(CP) & (AT), 197(HY), 198(ST), 199(EM), D-134(HY), D-141(HY).

JULY: [a] 200(SM)[b], 201(ST)[b], 202(EM)[b], 203(SM)[b], 204(AT)[b], 205(EM)[b], 206(ST)[b], 207(SA)[b], 208(SA)[b], 209(ST)[b], 210(SU)[b], 211(EM)[b], 212(SU)[b], 213(IR)[b], 214(HW)[b], 215(SM)[b], 216(ST)[b], 217(ST)[b], 218(ST)[b], 219(ST)[b], 220(SM)[b], 221(HW)[b], 222(SM)[b], 223(EM)[b], 224(EM)[b], 225(EM)[b], 226(CO)[b], 227(SM)[b], 228(SM)[b], 229(IR)[b].

AUGUST: 230(HY), 231(SA), 232(SA), 233(AT), 234(HW), 235(HW), 237(AT), 238(WW), 239(SA), 240(IR), 241(AT), 242(IR), 243(ST), 244(ST), 245(ST), 246(ST), 247(SA), 248(SA), 249(ST), 250(EM)[c], 251(ST), 252(SA), 253(AT), 254(HY), 255(AT), 256(ST), 257(SA), 258(EM), 259(WW).

SEPTEMBER: 260(AT), 261(EM), 262(SM), 263(ST), 264(WW), 265(ST), 266(ST), 267(SA), 268(CO), 269(CO), 270(CO), 271(SU), 272(SA), 273(PO), 274(HY), 275(WW), 276(HW), 277(SU), 278(SU), 279(SA), 280(IR), 281(EM), 282(SU), 283(SA), 284(SU), 285(CP), 286(EM), 287(EM), 288(SA), 289(CO).

OCTOBER: [d] 290(all Divs), 291(ST)[c], 292(EM)[c], 293(ST)[c], 294(PO)[c], 295(HY)[c], 296(EM)[c], 297(HY)[c], 298(ST)[c], 299(EM)[c], 300(EM)[c], 301(SA)[c], 302(SA)[c], 303(SA)[c], 304(CO)[c], 305(SU)[c], 306(ST)[c], 307(SA)[c], 308(PO)[c], 309(SA)[c], 310(SA)[c], 311(SM)[c], 312(SA)[c], 313(ST)[c], 314(SA)[c], 315(SM)[c], 316(AT), 317(AT), 318(WW), 319(IR), 320(HW).

NOVEMBER: 321(ST), 322(ST), 323(SM), 324(SM), 325(SM), 326(SM), 327(SM), 328(SM), 329(HW), 330(EM)[c], 331(EM)[c], 332(EM)[c], 333(EM)[e], 334(EM), 335(SA), 336(SA), 337(SA), 338(SA), 339(SA), 340(SA), 341(SA), 342(CO), 343(ST), 344(ST), 345(ST), 346(IR), 347(IR), 348(CO), 349(SM), 350(HW), 351(HW), 352(SA), 353(SU), 354(HY), 355(PO), 356(CO), 357(HW), 358(HY).

DECEMBER: 359(AT), 360(SM), 361(HY), 362(HY), 363(SM), 364(HY), 365(HY), 366(HY), 367(SU)[e], 368(WW)[e], 369(IR), 370(AT)[e], 371(SM)[e], 372(CO)[e], 373(ST)[e], 374(EM)[e], 375(EM), 376(EM), 377(SA)[e], 378(PO)[e].

VOLUME 80 (1954)

JANUARY: 379(SM)[e], 380(HY), 381(HY), 382(HY), 383(HY), 384(HY)[e], 385(SM), 386(SM), 387(EM), 388(SA), 389(SU)[e], 390(HY), 391(IR)[e], 392(SA), 393(SU), 394(AT), 395(SA)[e], 396(EM)[e], 397(ST)[e].

FEBRUARY: 398(IR)[f], 399(SA)[f], 400(CO)[f], 401(SM)[e], 402(AT)[f], 403(AT)[f], 404(IR)[f], 405(PO)[f], 406(AT)[f], 407(SU)[f], 408(SU)[f], 409(WW)[f], 410(AT)[f], 411(SA)[f], 412(PO)[f], 413(HY)[f].

MARCH: 414(WW)[f], 415(SU)[f], 416(SM)[f], 417(SM)[f], 418(AT)[f], 419(SA)[f], 420(SA)[f], 421(AT)[f], 422(SA)[f], 423(CP)[f], 424(AT)[f], 425(SM)[f], 426(IR)[f], 427(WW)[f].

APRIL: 428(HY)[e], 429(EM)[e], 430(ST), 431(HY), 432(HY), 433(HY), 434(ST).

MAY: 435(SM), 436(CP)[e], 437(HY)[e], 438(HY), 439(HY), 440(ST), 441(ST), 442(SA), 443(SA).

---

a. Beginning with "Proceedings-Separate No. 200," published in July, 1953, the papers were printed by the photo-offset method.
b. Presented at the Miami Beach (Fla.) Convention of the Society in June, 1953.
c. Presented at the New York (N.Y.) Convention of the Society in October, 1953.
d. Beginning with "Proceedings-Separate No. 290," published in October, 1953, an automatic distribution of papers was inaugurated, as outlined in "Civil Engineering," June, 1953, page 66.
e. Discussion of several papers, grouped by divisions.
f. Presented at the Atlanta (Ga.) Convention of the Society in February, 1954.